Output–Only Parametric Identification of a Scale Cable–Stayed Bridge Structure: a comparison of vector AR and stochastic subspace methods

Fotis P. Kopsaftopoulos, Panagiotis G. Apostolellis and Spilios D. Fassois*
University of Patras, Department of Mechanical & Aeronautical Engineering, Patras, Greece
URL: http://www.smsa.upatras.gr

ABSTRACT: A comparative assessment of two time–domain output–only vector structural identification methods, namely a Vector AutoRegressive (VAR) and a stochastic subspace method, is presented via their application to a laboratory cable–stayed bridge structure. A brief overview of the estimation methods is provided, while model order selection and validation are discussed. The modal frequencies and damping ratios are extracted and compared to those obtained via classical non–parametric techniques, while the methods' performance characteristics are assessed. The results highlight each method's facets and demonstrate how each one may be used for effective output–only identification.

1 INTRODUCTION

Structural identification under unobservable excitation is important in a large number of cases where the excitation is not measurable (Lardies and Ta 2011, Magalhães *et al.* 2009, Papakos and Fassois 2003, Petsounis *et al.* 2001). Typical examples include in–flight testing of aeronautical structures, in–operation testing of surface vehicle (automobile, railway) structures, as well as the testing of civil structures under ambient or seismic excitation. In such cases the identification has to be exclusively based upon the measured vibration responses.

Due to a number of advantages (such as reduced acquisition and analysis times, improved estimation accuracy, modal parameter "consistency"), *vector* methods, that is methods simultaneously accounting for several measured vibration signals, are most significant. Nevertheless, despite the progress achieved so far, our understanding of the methods' relative merits and performance characteristics as related to structural identification appears somewhat limited. This is, at least in part, due to the lack of comparative studies and critical assessments of the methods' pros and cons under various testing conditions.

The *goal* of this study is to contribute to filling this gap by presenting the application and experimental assessment of two important time–domain methods, namely a Vector AutoRegressive (VAR) and a stochastic subspace method using state space (SS) models, to a laboratory cable–stayed bridge structure. It should be emphasized that although Vector AutoRegressive Moving Average (VARMA) methods (Papakos and Fassois 2003) are more appropriate to their VAR counterparts and offer a better complement to SS methods, a VAR method is presently considered mainly due to simplicity – VARMA results are expected to be presented in a forthcoming paper.

*Corresponding author. Email: fassois@mech.upatras.gr; Tel/Fax: (++30) 2610 969 495 (direct); 969 492 (central).

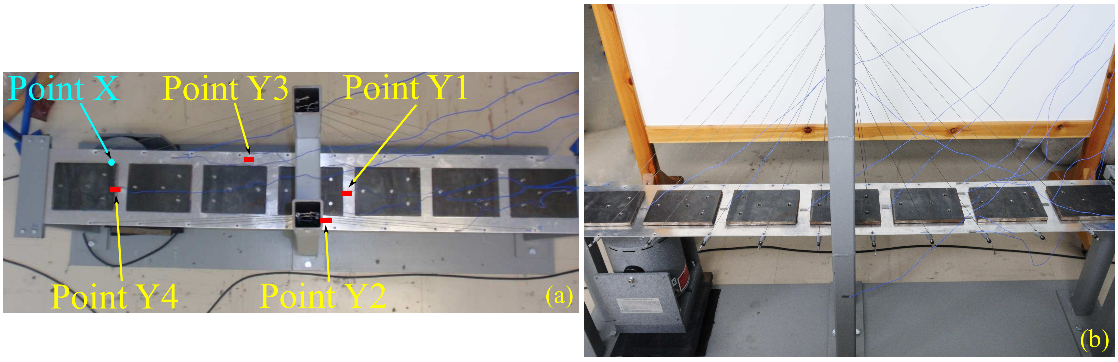

Figure 1: The scale cable–stayed bridge structure and the experimental set–up: (a) top–to–bottom view, and (b) general overview (force excitation at Point X and vibration measurement at Points Y1–Y4).

While in recent years both the VAR (Papakos and Fassois 2003, Petsounis *et al.* 2001) and SS (Lardies and Ta 2011, Magalhães *et al.* 2009, Weng *et al.* 2008) based structural identification frameworks have been frequently employed, it appears that no mutually comparative assessments are available.

Some of the main issues this study addresses between estimated VAR and stochastic SS models include: (i) achieved model parsimony (including required overdetermination), (ii) achieved model optimality (in terms of predictive ability and the Bayesian Information Criterion, BIC, Ljung 1999 pp. 505–507), (iii) identified structural dynamics (spectra and cross spectra) and modal parameter accuracy (point and interval estimates of modal quantities, missed modes, pseudo modes).

2 THE STRUCTURE AND THE EXPERIMENTAL SET–UP

The laboratory cable–stayed bridge structure and the test rig are shown in Fig. 1. The bridge deck is represented by a $1470 \times 190 \times 2$ mm aluminum plate suspended via $10 + 10$ cables attached to the central steel pylon and clamped to each edge of the deck. Seven $200 \times 120 \times 5$ mm steel plates are placed on the aluminum deck for increasing its mass. The excitation signal is zero–mean broadband random stationary Gaussian force applied vertically on the deck at Point X via an electromechanical shaker (MB Dynamics Modal 50A, max load 225 N) and measured via an impedance head (PCB 288D01, sensitivity 98.41 mV/lb), while the vibration responses at Points Y1–Y4 (Fig. 1) are measured via dynamic strain gauges (PCB 740B02, longitudinal orientation, $0.005 - 100$ kHz, 50 mV/$\mu\varepsilon$; sampling frequency $f_s = 256$ Hz, signal bandwidth $0.5 - 100$ Hz – see Table 1). The force excitation and strain response signals are driven through conditioning charge amplifiers (PCB 482A20 and PCB 481, respectively) into the data acquisition system consisting of two SigLab 20–42 measurement modules. The sample mean is subtracted from each signal, and scaling by the signal's sample standard deviation is implemented.

Table 1: Vibration signal characteristics and estimation method details.

Vibration signals	Signal bandwidth: $0.5 - 100$ Hz, Sampling frequency: $f_s = 256$ Hz Signal length (samples): $N = 23\,040$
Non–parametric (Welch) estimation	Segment length (samples): $L = 2\,048$, Frequency resolution: $\Delta f = 0.125$ Hz Window: Hamming, Overlap: 80% (MATLAB function: pwelch.m)
VAR estimation	Weighted Least Squares (WLS; single iteration) – QR implementation (MATLAB function arx.m)
SS estimation	Subspace CVA method – QR implementation (MATLAB function n4sid.m)

3 THE STRUCTURAL IDENTIFICATION METHODS

3.1 The Vector AR identification method

Under the standard assumption of uncorrelated excitation, the observed s–dimensional vibration (displacement, velocity, or acceleration) signal, say[1] $\boldsymbol{y}[t]$, may be modeled as an s–variate (presently $s = 4$) Vector AutoRegressive (VAR(n)) process of the form[2] (Fassois 2001, Ljung 1999):

$$\boldsymbol{y}[t] + \sum_{i=1}^{n} \boldsymbol{A}_i \cdot \boldsymbol{y}[t-i] = \boldsymbol{e}[t] \qquad E\{\boldsymbol{e}[t] \cdot \boldsymbol{e}^T[t]\} = \boldsymbol{\Sigma} \tag{1}$$

with $\boldsymbol{A}_i$ $(s \times s)$ designating the i–th AR matrix, $\boldsymbol{e}[t]$ $(s \times 1)$ the model residual (one–step–ahead prediction error) sequence characterized by the non–singular (and generally non–diagonal) covariance matrix $\boldsymbol{\Sigma}$, n the AR order, and $E\{\cdot\}$ statistical expectation.

Given the vibration signal measurements $\boldsymbol{y}[t]$ $(t = 1, 2 \ldots, N)$, the estimation of the VAR parameter vector $\boldsymbol{\theta}$ comprising all AR matrix elements ($\boldsymbol{\theta} = \text{vec}([\boldsymbol{A}_1 \ \ldots \ \boldsymbol{A}_n])$ and the residual covariance matrix $\boldsymbol{\Sigma}$ is accomplished via linear regression schemes based on minimization of the Ordinary Least Squares (OLS) or the Weighted Least Squares (WLS) criterion (Fassois 2001, Ljung 1999 p. 206).

The modeling procedure involves the successive fitting of VAR(n) models for increasing AR order n, until an adequate model is achieved. Model adequacy is checked via a combination of tools, which include monitoring of the Bayesian Information Criterion (BIC) (Ljung 1999 pp. 505–507) and the trace of the estimated residual covariance matrix $\widehat{\boldsymbol{\Sigma}}$ for a minimum value, as well as the use of frequency stabilization diagrams (Fassois 2001) which depict the evolution of estimated natural frequencies with increasing order. Their basis for structural mode distinction lies with the expectation that structural frequencies tend to "stabilize" (remain invariant) as the order increases, whereas "extraneous" frequencies change "randomly" within the considered frequency range.

3.2 The stochastic subspace (State Space, SS) identification method

The stochastic output–only linear multivariate (vector) state space model is of the form (Ljung 1999 Sec. 4.3, Van Overschee and De Moor 1996):

$$\begin{aligned} \boldsymbol{z}[t+1] &= \boldsymbol{A} \cdot \boldsymbol{z}[t] + \boldsymbol{K} \cdot \boldsymbol{e}[t] \\ \boldsymbol{y}[t] &= \boldsymbol{C} \cdot \boldsymbol{z}[t] + \boldsymbol{e}[t] \qquad E\{\boldsymbol{e}[t] \cdot \boldsymbol{e}^T[t]\} = \boldsymbol{\Sigma} \end{aligned} \tag{2}$$

with $\boldsymbol{y}[t]$ representing the s–dimensional vibration response vector, $\boldsymbol{z}[t]$ the p–dimensional state vector, $\boldsymbol{e}[t]$ an s–dimensional Gaussian zero–mean white vector sequence with covariance $\boldsymbol{\Sigma}$, and $\boldsymbol{A}, \boldsymbol{C}$ and $\boldsymbol{K}$ the system matrices. The order p of the system is the dimension of the state vector $\boldsymbol{z}[t]$, while $\boldsymbol{A} \in \mathbb{R}^{p \times p}$ is the dynamical system matrix. The matrix pair $\{\boldsymbol{A}, \boldsymbol{C}\}$ is assumed to be observable, which implies that all modes in the system can be observed in the output vector $\boldsymbol{y}[t]$.

The estimation of the unknown system matrices is presently achieved via subspace identification using the Canonical Variate Algorithm (CVA) (Ljung 1999 Sec. 10.6, Van Overschee and De Moor 1996 pp. 80–81). The modeling procedure involves the successive fitting of state space models for increasing order n, until an adequate model is selected (Fassois 2001, Ljung 1999). Model order selection is based on the BIC, monitoring the logarithm of singular values of the Hankel matrix obtained via Singular Value Decomposition (SVD) (Ljung 1999 Sec. 10.6), and the use of frequency stabilization diagrams (Fassois 2001).

[1] $t = 1, 2, \ldots$ indicates discrete time with the corresponding analog being $t \cdot T_s$ (T_s the sampling period).

[2] Bold face lower/upper case characters indicate vector/matrix quantities, respectively.

Table 2: Non–parametric (Welch based), VAR(49), and SS(55) modal parameter (point) estimates.

Non–parametric		VAR(49)		SS(55)		Non–parametric		VAR(49)		SS(55)	
Mode	f_n (Hz)	f_n (Hz)	ζ (%)	f_n (Hz)	ζ (%)	Mode	f_n (Hz)	f_n (Hz)	ζ (%)	f_n (Hz)	ζ (%)
$1^\dagger$	20.75	21.02	6.29	20.69	6.73	$9^\dagger$	53.25	53.43	2.01	54.01	2.67
	–	29.13	5.65	–	–	$10^\dagger$	54.75	54.71	0.76	54.52	0.51
$2^\dagger$	34.5	34.26	4.11	35.21	4.85		–	55.62	1.89	–	–
$3^\dagger$	36.5	36.01	3.05	36.90	0.41	$11^\dagger$	57.25	57.39	2.39	57.05	4.64
$4^\dagger$	38.5	37.79	1.76	38.22	1.41		–	60.60	2.39	–	–
	–	39.22	3.06	38.51	5.98	$12^\dagger$	61.5	61.54	0.60	61.41	0.33
$5^\dagger$	41.5	41.79	2.09	41.19	1.72	13	62.75	63.18	2.83	62.83	2.89
	–	42.75	5.12	42.21	4.04		–	67.98	2.73	–	–
$6^\dagger$	43.125	43.52	2.51	–	–	$14^\dagger$	69.75	70.06	1.79	69.95	2.40
7	46.625	45.53	2.90	46.67	6.88	15	73.125	72.61	1.05	72.66	0.85
$8^\dagger$	47.25	47.58	2.89	–	–	16	74.75	74.84	1.36	74.78	1.37
	–	–	–	50.86	0.50	17	76.5	77.47	3.36	–	–
	–	51.90	2.67	–	–		–	85.42	1.62	–	–
						18	95.5	95.59	2.29	–	–

†indicates mode evident in the non–parametric (Welch based) spectral estimates.

4 STRUCTURAL IDENTIFICATION RESULTS

The structural identification methods are now assessed via their application to a scale cable–stayed laboratory bridge structure. The structure exhibits 18 modes in the $0.5 - 100$ Hz range, the validity of which has been confirmed via input–output analysis using both non–parametric and parametric (VAR with eXogenous excitation, VARX) techniques (not presented). A maximum damping ratio of 10% has been set for distinguishing structural from "extraneous" modes.

All identification results presented in the sequel are based on $N = 23\,040$ (≈ 90 s) sample–long response signals obtained from the *four* vibration measurement locations on the structure (Fig.1).

4.1 *Non–parametric identification results*

An $L = 2\,048$ sample–long Hamming data window with 80% overlap is used for Welch–based spectral estimation (Table 1; MATLAB function *pwelch.m*). The obtained Power Spectral Density (PSD) estimates for all vibration measurement locations are depicted in Fig. 2. There are 18 modes included in the bandwidth of $0.5 - 100$ Hz; nevertheless 12 modes are most prominent in the non–parametric spectra (see Fig. 2 and Table 2).

4.2 *VAR identification results*

The successive fitting of 4–variate VAR(n) models for $n = 2, \ldots, 80$ (MATLAB function *arx.m*) leads to decreasing BIC, which reaches an approximate plateau for model order $n > 40$, while BIC minimization is achieved for order $n = 49$ (Fig. 3).

Moreover, as indicated in the frequency stabilization diagram of Fig. 4a, model orders of $n > 40$ are adequate for most natural frequencies to get stabilized. Notice the color bar in the figure, which presents the damping ratio range. Two norms of the estimated residual covariance matrices ($\widehat{\boldsymbol{\Sigma}}$) are presented in Fig. 5 for increasing model order – these are essentially measures of the models' optimality (in terms of predictive ability).

Model parsimony is examined in Fig. 6: Fig. 6a depicts the Samples Per Parameter (SPP) ratio versus increasing model order p (lower horizontal axis) and the VAR order n (upper horizontal axis). Notice that the SPP ratio is sufficient even for the VAR(80) model, as well as that the model

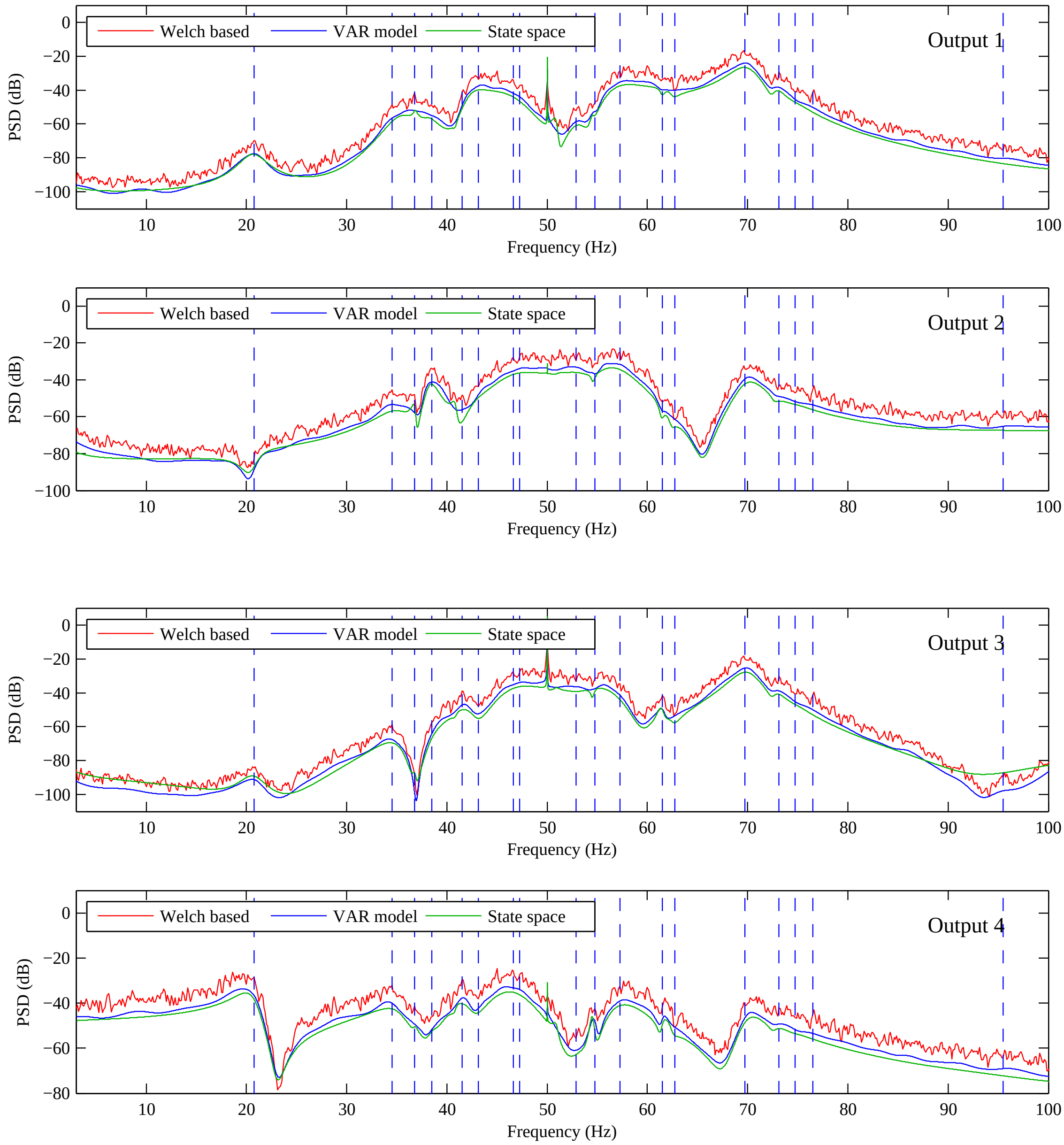

Figure 2: Non–parametric (Welch–based) and parametric (VAR(49) and SS(55) based) spectral estimates. The dashed vertical lines indicate actual structural modes.

order is four times the VAR order. Fig. 6b depicts the number of model parameters estimated versus the model order p (lower horizontal axis) and the VAR order n (upper horizontal axis).

The above identification procedure leads to the selection of a 4–variate VAR(49) model (Table 3). The identified VAR(49) representation has 784 parameters with the SPP being equal to 29.4. The VAR(49) based spectra for all vibration measurement locations are presented in Fig. 2, where they are contrasted to their non–parametric (Welch based) counterparts.

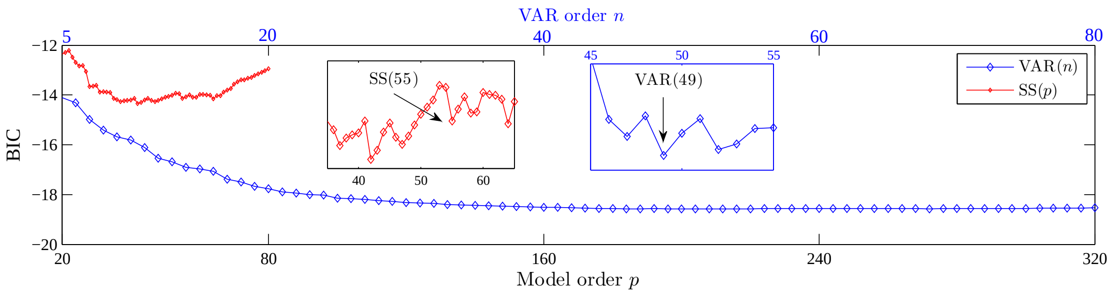

Figure 3: Model order selection: BIC criterion for VAR(n) and SS(p) type parametric models.

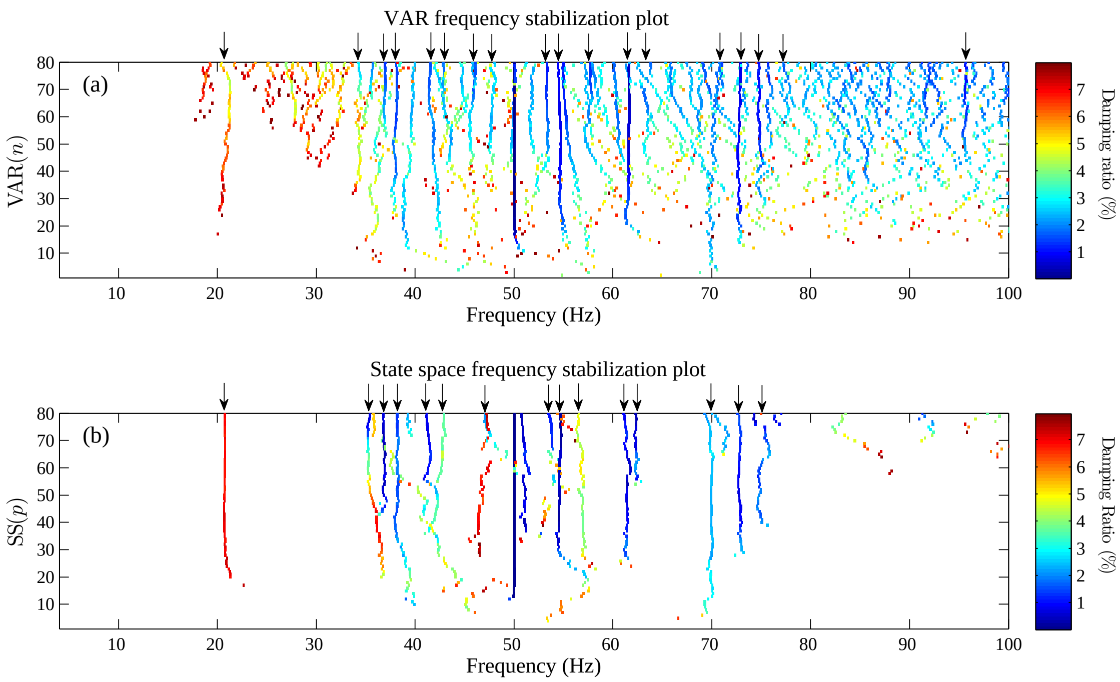

Figure 4: Frequency stabilization diagrams for VAR(n) and SS(p) type models.

4.3 Stochastic subspace (state space, SS) identification results

Stochastic subspace identification of the structural dynamics (MATLAB function *n4sid.m*) involves the successive fitting of 4–variate SS models for $p = 2, \ldots, 80$. This leads to decreasing BIC, which reaches a minimum for order $n = 42$ (Fig. 3b). Furthermore, as indicated by the frequency stabilization diagram of Fig. 4, model orders of $p > 50$ are needed for most natural frequencies to get stabilized.

The two norms of the estimated residual covariance matrices $(\widehat{\boldsymbol{\Sigma}})$ are presented in Fig. 5 for increasing model order – these are essentially measures of the models' predictive ability. Model parsimony is examined in Fig. 6: The Samples Per Parameter (SPP) versus increasing model order p (lower horizontal axis) is presented in Fig. 6a, while the number of model parameters versus model order is depicted in Fig. 6b. Notice that the SPP goes below 20 for orders $p > 30$, and decreases to reach only 3.24 for $p = 80$.

A 4–variate SS model of order $p = 55$ is finally selected as adequate (Table 3). The identified SS representation has $3\,640$ parameters with the SPP being equal to 6.33. The SS based spectra for all vibration measurement locations are presented in Fig. 2, where they are compared to their non–parametric (Welch based) and VAR(49) counterparts.

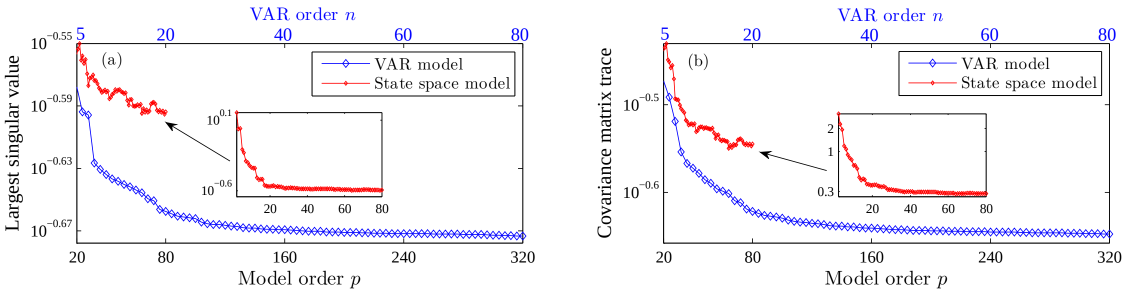

Figure 5: Model optimality in terms of predictive ability for VAR(n) and SS(p) type parametric models: (a) largest singular value of $\widehat{\Sigma}$ and (b) trace of $\widehat{\Sigma}$ for increasing model order.

4.4 Discussion

Both VAR and SS model based identified structural modes are in very good agreement between themselves, and also with the non–parametric analysis (see Tables 2, 3 and Fig. 2). The identified modes are mostly characterized by low to medium damping ratios, as ζ lies in the interval $[0.6\% - 6.3\%]$. The VAR model is characterized by increased overdetermination (515%) compared to the SS model (147%), which influences the number of identified spurious modes. This is also obvious from the frequency stabilization diagrams (Fig. 4), where the VAR based figure exhibits an increased number of non–stabilized (spurious) modes for higher model orders, whereas the SS model based figure exhibits a clearly smaller number. The VAR method effectively identifies all 18 structural modes, whereas the SS (subspace) method is able to identify only 14 (misses 4). Nevertheless, the VAR model exhibits 8 spurious modes, while the SS exhibits 2 (Table 2).

Furthermore, Fig. 6a shows that the SPP ratio is significantly higher for the VAR models compared to their SS counterparts, as the SS representations use a significantly higher number of estimated parameters (Fig. 6b and Table 3). This results in reduced statistical robustness, a claim enhanced by the fact that the SS covariance matrix of the estimated parameters cannot be computed (Table 3). Nevertheless, the low SPP ratio does not seem to have an effect on the SS based point modal estimates, as the corresponding results are very accurate. Yet it is clear that sufficiently long data records should be employed in order for the SS models to be effectively estimated. Finally, as Fig. 5 demonstrates and despite the lower parametrization (number of estimated parameters), the VAR models exhibit an overall better predictive ability than that of their SS counterparts (lower residual covariance matrix singular value and trace – Table 3). This may be viewed as somewhat natural, due to the fact that VAR models are estimated by minimizing a quadratic prediction error criterion.

Table 3: Estimated VAR(49) and SS(55) model characteristics.

		VAR(49)	SS(55)
Model Parsimony	Model order	196	55
	overdetermination $(\%)$	515%	147%
	Number of parameters	784	3 520
	Samples Per Parameters (SPP)	29.4	6.54
Model optimality	Norm of residual cov. matrix	0.21	0.26
(incl. predictive ability)	Trace of residual cov. matrix	0.23	0.29
	BIC	-18.57	-14.13
Modal parameter estimation	Point estimates	+ 18 estimated modes - 8 pseudo modes	+ 14 estimated modes - 2 pseudo modes
	Interval (uncertainty) estimates:		
	Condition number of estimated parameter covariance	5×10^8	n/a

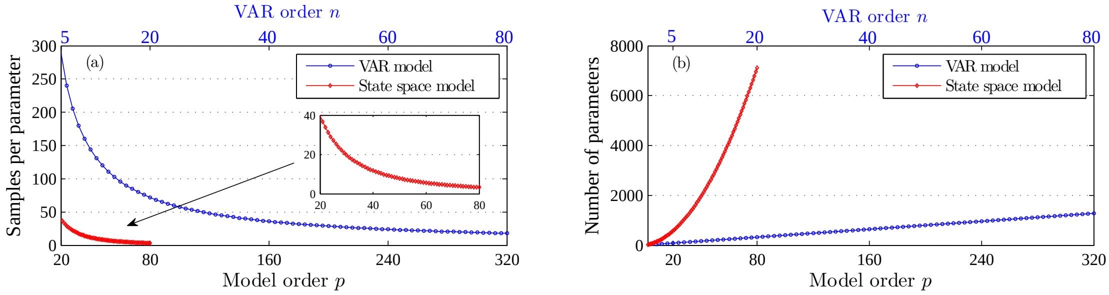

Figure 6: Model parsimony for VAR(n) and SS(p) type parametric models: (a) samples per parameter (SPP) and (b) number of parameters for increasing model order.

5 CONCLUDING REMARKS

The main conclusions drawn from this work may be summarized as follows:

(a) Both of the VAR and SS identification methods used are in good agreement between themselves and also with non–parametric (Welch based) analysis.

(b) The estimated VAR model – albeit of higher order – is far more parsimonious than its estimated SS counterpart (784 versus 3 520 parameters). This drastically increased number of parameters may cause statistical reliability problems for "short" signal records.

(c) The VAR model effectively identifies all 18 structural modes, while it exhibits a somewhat increased number of spurious modes (8) due to its increased degree of overdetermination (515%). The SS model fails to identify 4 structural modes, but exhibits only 2 spurious modes (degree of overdetermination 147%). Due to their smaller numbers, the distinction of spurious modes may be somewhat simpler in the SS model case, but, on the other hand, the method is more likely to miss some of the "weaker" structural modes.

(d) Although point modal estimates are similar in the VAR and SS cases, interval (uncertain) modal estimates could be obtained only in the VAR case. They could not be obtained in the SS case due to very poorly conditioned covariance matrix estimate.

(e) Further comparisons are necessary for fully assessing the methods – these should be expanded to include full VARMA models as well.

REFERENCES

Fassois, S.D. 2001. *Parametric identification of vibrating structures*. In S.G. Braun, D.J. Ewins, and S.S. Rao, editors, Encyclopaedia of Vibration, pp. 673–685, Academic Press.

Lardies, J. and Ta, M. 2011. Modal parameter identification of stay cables from output–only measurements, *Mech. Syst. and Signal Pr.*, **25**, pp. 133–150

Ljung, L. 1999. *System Identification: Theory for the User*. Prentice–Hall, 2nd edition.

Magalhães, F., Cunha, A. and Caetano, E. 2009. Online automatic identification of the modal parameters of a long span arch bridge, *Mech. Syst. and Signal Pr.*, **23**, pp. 316–329.

Van Overschee, P. and De Moor, B. 1996. *Subspace Identification for Linear Systems: Theory, Implementation, Applications*. Kluwer Academic Publishers.

Papakos, V. and Fassois, S.D. 2003. Multichannel identification of aircraft skeleton structures under unobservable excitation: A vector AR/ARMA framework. *Mech. Syst. and Signal Pr.*, **17**(6), pp. 1271–1290.

Petsounis, K.A., Rizos, D.A. and Fassois, S.D. 2001. In–operation vector autoregressive spectral and modal analysis of railway vehicle dynamics. *Proc. of 4th Eur. Conference on Noise Control*, Patras, Greece.

Weng, J., Loh, C., Lynch, J.P., Lu, K., Lin, P. and Wang, Y. 2008. Output–only modal identification of a cable–stayed bridge using wireless monitoring systems. *Eng. Struct.*, **30**, pp. 1820–1830.